Part III Category Theory

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Contents

1 Definitions and Examples

Definition 1.1 (Category)**.** A category $\mathcal{C}$ consists of

a. a collection $\mathrm{ob}\,\mathcal{C}$ of **objects** A, B, C, ...

b. a collection $\mathrm{mor}\,\mathcal{C}$ of **morphisms** f, g, h, ...

c. two operations dom, cod from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$'

d. an operation assigning to each object A a morphism $1_A : A \to A$

e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$ $\mathrm{dom}\, g = \mathrm{cod}\, f$, and then $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$

satisfying

f. $f1_A = f$ and $1_B f = f$ $\forall f : A \to B$

g. $h(fg) = (hg)f$ whenever gf and hg are defined

Definition 1.2 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $\mathcal{C} \to \mathcal{D}$ consists of

a. a mapping $A \to FA$ from $\mathrm{ob}\,\mathcal{C}$ to $\mathrm{ob}\,\mathcal{D}$

b. a mapping $f \to Ff$ from $\mathrm{mor}\,\mathcal{C}$ to $\mathrm{mor}\,\mathcal{D}$

satisfying $\mathrm{dom}\, Ff = F\mathrm{dom}\, f$, $\mathrm{cod}\, Ff = F\mathrm{cod}\, f$ for all f, $F(1_A) = 1_{FA}$ for all A, and $F(gf) = (Fg)(Ff)$ whenever gf is defined.

Definition 1.3. By a **contravariant functor** $\mathcal{C} \to \mathcal{D}$ we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or equivalently $\mathcal{C}^{op} \to \mathcal{D}$). A functor $\mathcal{C} \to \mathcal{D}$ is sometimes said to be **covariant**.

Definition 1.4 (Natural transformation)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \to G$ assigns to each $A \in \mathrm{ob}\,\mathcal{C}$ a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$, such that

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition 1.5. Given categories $\mathcal{C}, \mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

Lemma 1.6. *Given $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \to G$, α is an isomorphism in $[\mathcal{C}, \mathcal{D}] \iff$ each α_A is an isomorphism in $\mathcal{D}$.*

Definition 1.7 (Faithful and full)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

a. We say that F is **faithful** if, given $f, g \in \operatorname{mor} \mathcal{C}$, the equations $\operatorname{dom} f = \operatorname{dom} g$, $\operatorname{cod} f = \operatorname{cod} g$ and $Ff = Fg$ imply $f = g$.

b. F is **full** if, given any $g : FA \to FB$ in $\mathcal{D}$, there exists $f : A \to B$ in $\mathcal{C}$ with $Ff = g$.

c. We say a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

We say a property P of categories is **categorical** if whenever $\mathcal{C}$ has P and $\mathcal{C} \simeq \mathcal{D}$ then $\mathcal{D}$ has P.

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category)**.** Given an object B of a category $\mathcal{C}$, define the **slice category** $\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commute.

Lemma 1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and **essentially surjective**, i.e. for every $B \in \operatorname{ob} \mathcal{D}$, there exists $A \in \operatorname{ob} \mathcal{C}$ s.t. $FA \cong B$.*

Definition 1.11. a. A **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object from each isomorphism class of objects of $\mathcal{C}$.

b. We say $\mathcal{C}$ is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in $\mathcal{C}$ satisfies $\operatorname{dom} f = \operatorname{cod} f$.

For example, $\mathbf{Mat}_K$ is skeletal. The full subgategory of standard vector spaces K^n is a skeleton of **fd** $\mathbf{Mod}_K$.

Remark 1.12. The following statements are each equivalent to the Axiom of Choice:

1. Every small category has a skeleton
2. Any small category is equivalent to each of its skeletons
3. Any two skeletons of a given small category are isomorphic

Definition 1.13. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$.

a. f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.

b. Dually, f is an **epimorphism** if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.

c. $\mathcal{C}$ is a **balanced** category if every $f \in \operatorname{mor} \mathcal{C}$ which is both monic and epic is an isomorphism.

2 The Yoneda Lemma

Definition 2.1. A category $\mathcal{C}$ is **locally small** if, for any two objects A, B of $\mathcal{C}$, the morphism $A \to B$ are parametrised by a set $\mathcal{C}(A, B)$.

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: if $g : B \to B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \to B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \to \mathbf{Set}$.

Lemma 2.2 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob} \mathcal{C}$ and $F : \mathcal{C} \to$ **Set**. Then*

i. There is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.

ii. Moreover, this bijection is natural in both A and F.

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \to F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \to F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \to C$, we have

$$\begin{aligned}
\Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\
&= F(gf)(x) \\
&= (Fg)(Ff)(x) \\
&= (Fg)\Psi(x)_B(f)
\end{aligned}$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \to B$,

$$\begin{aligned}
\Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\
&= Ff(\alpha_A(1_A)) \\
&= \alpha_B(\mathcal{C}(A, f)(1_A)) \\
&= \alpha_B(f)
\end{aligned}$$

□

Corollary 2.3. *The mapping $A \to \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \textbf{\textit{Set}}]$.*

Proof. Given two objects A, B, 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \to B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ □

We call this functor the **Yoneda embedding**. Hence any locally small category $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$.

Definition 2.4. A functor $\mathcal{C} \to \mathbf{Set}$ is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A.

A **representation** of $F : \mathcal{C} \to \mathbf{Set}$ is a pair (A, x) where $A \in \operatorname{ob}\mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We also call x a **universal element** of F.

Corollary 2.5 ('Representations are unique up to unique isomorphism')**.** *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \to \textbf{\textit{Set}}$, then there's a unique isomorphism $f : A \to B$ s.t $Ff(x) = y$.*

Definition 2.6 (Product and coproduct)**.** Given two objects A, B of a locally small category $\mathcal{C}$, we define their **product** to be a representation of the functor

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \to \mathbf{Set}$$

i.e. an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ s.t. given any pair $(f : C \to A, g : C \to B)$, there exists a unique $h : C \to A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.

More generally, we can define the product $\prod_{i \in I} A_i$ of a family $\{A_i \mid i \in I\}$ of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every A there's a unique $A \to 1$.

Dualizing, we get the notion of **coproduct** or **sum**.

Definition 2.7 (Equaliser and coequaliser)**.** Given a parallel pair $f, g : A \rightrightarrows B$ in a locally small category $\mathcal{C}$, the assignment $C \mapsto FC = \{h : C \to A \mid fh = gh\}$ is a subfunctor F of $\mathcal{C}(-, A)$. A representation of F is called an **equaliser** of (f, g).

In elementary terms, it's an object E equipped with $e : E \to A$ s.t. $fe = ge$, s.t. any h with $fh = gh$ factors uniquely as $h = ek$

Dually, we have the notion of **coequaliser**, i.e. a morphism $q : B \to Q$ satisfying $qf = qg$, and universal among such.

Definition 2.8. a. We say a monomorphism is **regular** if it occurs as an equaliser (dually, regular epimorphism).

b. We say $f : A \to B$ is a **split monomorphism** if there exists $g : B \to A$ with $gf = 1_A$.

Every split monomorphism is regular: if $gf = 1_A$, f is an equaliser of $(1_B, fg)$ [see sheet 1, q2].

Definition 2.9. Let $\mathcal{C}$ be a (locally small) category, $\mathcal{G}$ a collection of objects of $\mathcal{C}$.

a. Say $\mathcal{G}$ is a **separating family** if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are jointly faithful, i.e. if given $f, g : A \rightrightarrows B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \to A$ with $fh \neq gh$.

b. Say $\mathcal{G}$ is a **detecting family** if the $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ jointly reflect isomorphisms, i.e. if given $f : A \to B$ s.t. every $g : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, f is an isomorphism.

Lemma 2.10. *i. If $\mathcal{C}$ is balanced, then any separating family is detecting* *ii. If $\mathcal{C}$ has equalisers, then every detecting family is separating*

Definition 2.11. An object P is **projective** if $\mathcal{C}(P, -)$ preserves epimorphisms, i.e. if given

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

there exists $g : P \to A$ with $eg = f$.

Dually, P is **injective** in $\mathcal{C}$ if it's projective in $\mathcal{C}^{op}$.

If P satisfies this property $\forall e$ in some class $\mathcal{E}$ of epimorphisms, we call it $\mathcal{E}$-projective.

Corollary 2.12. *Representable functors are (pointwise) projective in* $[\mathcal{C}, \boldsymbol{Set}]$

Proof. Given

$$\begin{tikzcd} & \mathcal{C}(A, -) \arrow[d, "\beta"] \\ F \arrow[r, two heads, "\alpha"] & G \end{tikzcd}$$

β corresponds to some $y \in GA$. α_A is surjective, so $\exists x \in FA$ with $\alpha_A(x) = y$. x corresponds to $\gamma : \mathcal{C}(A, -) \to F$ with $\alpha\gamma = \beta$. $\square$

3 Adjunctions

Definition 3.1 (D.M. Khan, 1958)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two functors. An **adjunction** between F and G is a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morphisms $A \to GB$ in $\mathcal{C}$, which is natural in A and B.

(If $\mathcal{C}$ and $\mathcal{D}$ are locally small, this says that $(A, B) \to \mathcal{D}(FA, B)$ and $(A, B) \to \mathcal{C}(A, GB)$ are naturally isomorphic functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$).

We say F is **left adjoint** to G, or G is **right adjoint** to F, and write $F \dashv G$.

Theorem 3.2. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Given $A \in \operatorname{ob} \mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \operatorname{ob} \mathcal{D}$, $f : A \to GB$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ in $\mathcal{D}$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.

Proof. First suppose G has a left adjoint F. Let $\eta_A : A \to GFA$ be the morphism corresponding to $1_{FA} : FA \to FA$. The pair (FA, η_A) is an object of $(A \downarrow G)$. We'll show it's initial.

Given $g : FA \to B$, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ must correspond to $FA \xrightarrow{1} FA \xrightarrow{g} B$ under the adjunction.

So, for any object (B, f) of $(A \downarrow G)$, the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$ is the morphism $FA \to B$ corresponding to f.

Conversely, suppose we're given an initial object (FA, η_A) of $(A \downarrow G)$ for each G. Given $f : A \to A'$, the composite $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$ is an object of $(A \downarrow G)$, so there's a unique morphism $Ff : FA \to FA'$ making

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{a'}"] & GFA' \end{tikzcd}$$

commute.

$f \mapsto Ff$ is functorial: given $f' : A' \to A''$, then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$, so they're equal.

Finally, given $f : A \to GB$, the morphism $g : FA \to B$ corresponding to it is the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$.

The naturality of this bijection is given by naturality of η, and naturality in B is immediate. $\square$

Corollary 3.3. *If F, F' are both left-adjoint to G, then there's a canonical natural isomorphism $F \to F'$.*

Proof. For each A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \to (F'A, \eta'_A)$.

α is natural: given $f : A \to A'$, $\alpha_{A'} f$ and $(Ff)\alpha_A$ are both morphisms $(FA, \eta_A) \to (F'A', \eta'_{A'} f)$ in $(A \downarrow G)$. So they're equal. $\square$

Lemma 3.4. *Given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$ *, if $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.*

Proof. We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in A and C. $\square$

Corollary 3.5. *Given a commutative square* $\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r, "K"] & \mathcal{F} \end{tikzcd}$ *of categories and functors, suppose all the functors in the diagram have left adjoints. Then the diagram* $\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{C} \end{tikzcd}$ *of left adjoints commutes up to natural isomorphism.*

Given $F \dashv G$, we have a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ defined as in 3.2. We call η the **unit** of the adjunction.

Dually, we have $\epsilon : FG \to 1_{\mathcal{D}}$, the **counit**. $\epsilon_B : FGB \to B$ corresponds to $1_{GB} : GB \to GB$.

Theorem 3.6. *Suppose we're given* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$*. Specifying an adjunction* $F \dashv G$ *is equivalent to specifying natural transformations* $\eta : 1_{\mathcal{C}} \to GF$ *and* $\epsilon : FG \to 1_{\mathcal{D}}$ *such that*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

commute. (We say η *and* ϵ *satisfy the* ***triangular identities****).*

Proof. Given $F \dashv G$, we define η and ϵ as already described. Since $\epsilon_{FA} : FGFA \to FA$ corresponds to 1_{GFA}, the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$, so it must be 1_{FA}.

Similarly for the other identity.

Conversely, given η and ϵ satisfying the $\triangle^r$ identities, we map $f : A \to GB$ to the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$ and $g : FA \to B$ to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$.

We have

$$\Phi(\ A \xrightarrow{f} GB\ ) = \ FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$$

$$\Psi(\ FA \xrightarrow{g} B\ ) = \ A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$$

So

$$\begin{aligned} \Psi\Phi(f) &= \ A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= \ A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f \end{aligned}$$

And dually $\Phi\Psi(g) = g$.

Naturality of Φ in A is immediate from its definition, and naturality in B follows from that of ϵ. $\square$

Lemma 3.7. *Suppose given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *and natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. *Then there exist natural isomorphisms* α', β' *which additionally satisfy the triangular identities. In particular* $(F \dashv G)$.

Proof. We define $\alpha' = \alpha$ and take β' to be the composite

$$FG \xrightarrow{\beta^{-1}_{FG}} FGFG \xrightarrow{F^{-1}_{\alpha_G}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since
$$\begin{tikzcd} FGFG \arrow[r, "FG\beta"] \arrow[d, "\beta_{FG}"'] & FG \arrow[d, "\beta"] \\ FG \arrow[r, "\beta"] & 1_{\mathcal{D}} \end{tikzcd}$$
commutes and β is monic, we have $FG\beta = \beta_F G$.

Similarly, $GF\alpha = \alpha_{GF} : GF \to GFGF$.

Now

$$\begin{aligned} \beta'_F \circ F_{\alpha'} = &\; F \xrightarrow{F\alpha} FGF \xrightarrow{\beta^{-1}_{FGF}} FGFGF \xrightarrow{F\alpha^{-1}_{GF}} FGF \xrightarrow{\beta_F} F \\ = &\; F \xrightarrow{\beta^{-1}_F} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F \\ = &\; 1_F \end{aligned}$$

and

$$\begin{aligned} G\beta' \circ \alpha'_G = &\; G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{GF^{-1}_{\alpha_G}} GFG \xrightarrow{G\beta} G \\ = &\; G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha^{-1}_{GFG}} GFG \xrightarrow{\beta_F} G \\ = &\; 1_G \end{aligned}$$

$\square$

Lemma 3.8. *Suppose* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $(F \dashv G)$ *is an adjunction with counit* ϵ. *Then*

i. ϵ *is (pointwise) epic* $\iff$ G *is faithful*

ii. ϵ *is an isomorphism* $\iff$ G *is full and faithful*

Proof. i. Given $g : B \to B'$, the morphism $Gg : GB \to GB'$ corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

So, for fixed B, composition with ϵ_B is injective on morphisms $B \to B'$ $\iff$ $(g \mapsto Gg)$ is injective on morphisms $B \to B'$.

Hence G is faithful $\iff$ ϵ_B is epic $\forall B$.

ii. Similarly, ϵ_B is 0 $\forall B \implies G$ is bijective on morphisms with given domain and codomain, i.e. G is full and faithful.

Conversely, if G is full and faithful, 1_{FGB} factors uniquely as $FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$, so ϵ_B is split monic. But it's epic by (i), hence an isomorphism.

□

Definition 3.9. i. A **reflection** is an adjunction satisfying the conditions of 3.8(ii).

ii. A **reflective** subcategory of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.

Dually, **coreflection** and **coreflective** subcategory.

4 Limits

Definition 4.1. a. Let J be a category (almost always small, often finite). A **diagram of shape J** in a category $\mathcal{C}$ is a functor $D : J \to \mathcal{C}$.

E.g. if J is the finite category
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$
, a diagram of shape J is a commutative square. If J is the category
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$
, a diagram of shape J is a not-necessarily-commutative square.

The objects $D(j)$, $j \in \operatorname{ob} J$ are called **vertices** of D, and the morphisms $D(\alpha)$, $\alpha \in \operatorname{mor} J$ are called **edges** of D.

b. Let $D : J \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A **cone over D** is a pair $(A, (\lambda_j \mid j \in \operatorname{ob} J))$ where $\lambda_j : A \to D(j)$ $\forall j$, and
$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"] & & D(j') \end{tikzcd}$$
commutes for each $\alpha : j \to j'$ in J.

A is called the **apex** of the cone, and the λ_j are its **legs**.

Equivalently, λ is a natural transformation $\triangle A \to D$, where $\triangle A$ is the **constant diagram** with all vertices A and all edges 1_A.

A **morphism** $f : (A, (\lambda_j)) \to (B, (\mu_j))$ of cones over D is a morphism $f : A \to B$ s.t.
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda_j"'] & & B \arrow[dl, "\mu_j"] \\ & D(j) & \end{tikzcd}$$
commutes for each j. We have a category **Cone**(D) of cones over D.

Note that $A \mapsto \triangle A$ is a functor $\mathcal{C} \to [J, \mathcal{C}]$ and **Cone**(D) is in fact the category $(\triangle \downarrow D)$.

A cocone over $D : J \to \mathcal{C}$ is a cone over $D : J^{op} \to \mathcal{C}^{op}$. We write **Cocone**$(D)$ for the category of cocones over D.

Definition 4.2. i. A **limit** (resp. **colimit**) for a diagram $D : J \to \mathcal{C}$ is a terminal object of **Cone**(D) (respectively an initial object of **Cocone**(D)).

ii. We say $\mathcal{C}$ has limits (resp. colimits) of shape J if $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right (resp. left) adjoint.

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape J).

Definition 4.3 (Pullback)**.** Let J be $\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$. A diagram of shape J looks like $\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$. A cone over it consists of $\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, "l"] & A \\ C & B \end{tikzcd}$ satisfying $fh = l = gk$. Equivalently, it's a pair $\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] & A \\ C & \end{tikzcd}$ completing the diagram to a commutative square.

A universal such pair is called a **pullback** (or **fibre product**); in **Set** it can be defined as $\{(a, b) \in A \times B \mid f(a) = g(b)\}$. A colimit of shape J^{op} is called a **pushout**.

Theorem 4.4. *Let $\mathcal{C}$ be a category.*

i. If $\mathcal{C}$ has equalisers and all finite (resp. all small) products, then $\mathcal{C}$ has all finite (resp. all small) limits.

ii. If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.

Proof. i. Given $D : J \to \mathcal{C}$, first form the products

$$P = \prod_{j \in \text{ob}\, J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \text{mor}\, J} D(\text{cod}\, \alpha)$$

Define $P \overset{f}{\underset{g}{\rightrightarrows}} Q$ by $\pi_\alpha f = \pi_{\operatorname{cod}\alpha} : P \to D(\operatorname{cod}\alpha)$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\operatorname{dom}\alpha} : P \to D(\operatorname{dom}\alpha) \to D(\operatorname{cod}\alpha)$, and let $e : E \to P$ be the equaliser of (f, g).

Claim $(E, (\pi_j e \mid j \in \operatorname{ob} J))$ is a limit cone for D. It is a cone since, for any $\alpha : j \to j'$, $D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e$.

Given any cone $(C, (\lambda_j \mid j \in \operatorname{ob} J))$, the λ_j define a unique $\lambda : C \to P$, and $f\lambda = g\lambda$ since $\pi_\alpha f \lambda = \pi_\alpha g \lambda \ \forall \alpha$. So λ factors uniquely through e.

ii. Let 1 be a terminal object of $\mathcal{C}$. For any pair of objects (A, B) the pullback of

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

has the universal property of a product $A \times B$, so $\mathcal{C}$ has binary products. Then we can define any finite product $\prod_{i=1}^n A_i$ as $(((A_1 \times A_2) \times A_3) \times \dots) \times A_n$.

So we need to show $\mathcal{C}$ has equalisers. Given $A \overset{f}{\underset{g}{\rightrightarrows}} B$, consider the pullback of

$$\begin{tikzcd} & B \arrow[d, "(1_A, f)"] \\ A \arrow[r, "(1_A, g)"] & A \times B \end{tikzcd}.$$

It consists of

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "k"] & B \\ A & \end{tikzcd}$$

satisfying $1_A h = 1_A k$ and $fh = gk$, and universal among such.

But this forces $h = k$, and h has the universal property of an equaliser for (f, g). So by (i), $\mathcal{C}$ has all finite limits.

□

Definition 4.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

a. We say F **preserves** limits of shape J if, given $D : J \to \mathcal{C}$ and a limit cone $(L, (\lambda_j : j \in \operatorname{ob} J))$ for D, the cone $(FL, (F\lambda_j : j \in \operatorname{ob} J))$ is a limit for $FD : J \to \mathcal{D}$.

b. We say F **reflects** limits of shape J if, given $D : J \to \mathcal{C}$ and a cone $(L, (\lambda_j))$ such that $(FL, (F\lambda_j))$ is a limit for FD, then $(L, (\lambda_j))$ is a limit for D.

c. We say F **creates** limits of shape J if, given $D : J \to \mathcal{C}$ and a limit $(M, (\mu_j))$ for FD, there exists a cone (L, λ_j) over D whose image is isomorphic to $(M, (\mu_j))$, and any such cone is a limit for D.

Lemma 4.6. *Suppose $\mathcal{D}$ has limits of shape J. Then $[\mathcal{C}, \mathcal{D}]$ has limits of shape J, and they're constructed pointwise (i.e. the forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\operatorname{ob}\mathcal{C}}$ creates them).*

Proof. Consider a functor $D : J \times \mathcal{C} \to \mathcal{D}$. For each $A \in \operatorname{ob}\mathcal{C}$, let $(LA, (\lambda_{j,A} : LA \to D(j, A) \mid j \in \operatorname{ob} J))$ be a limit for the diagram $D(-, A) : J \to \mathcal{D}$.

Given any $f : A \to B$ in $\mathcal{C}$, the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, so they induce a unique $Lf : LA \to LB$ such that

$$\begin{array}{ccc} LA & \xrightarrow{Lf} & LB \\ \downarrow{\scriptstyle \lambda_{j,A}} & & \downarrow{\scriptstyle \lambda_{j,B}} \\ D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \end{array}$$

commutes for all j. Uniqueness assures $L(gf) = L(g)L(f)$, so L is a functor $\mathcal{C} \to \mathcal{D}$, and the $\lambda_{j,-}$ are natural transformations $L \to D(j, -)$.

Suppose we're given any cone over D in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $\mu_j : M \to D(j, -)$. Then $(MA, (\mu_{j,A} : MA \to D(j, A) \mid j \in \operatorname{ob} J))$ is a cone over $D(-, A)$ in $\mathcal{D}$, so we get a unique $\nu_A : MA \to LA$ s.t. $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j.

Uniqueness tells us that

$$\begin{array}{ccc} MA & \xrightarrow{Mf} & MB \\ \downarrow{\scriptstyle \nu_A} & & \downarrow{\scriptstyle \nu_B} \\ LA & \xrightarrow{Lf} & LB \end{array}$$

commutes for all $f \in \operatorname{mor}\mathcal{C}$, so $\nu : M \to L$ in $[\mathcal{C}, \mathcal{D}]$, so it's the unique factorisation of the $\mu_{j,-}$ through the $\lambda_{j,-}$. $\square$

Lemma 4.7. *A morphism $f : A \to B$ is monic $\iff$*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

Proof. f is monic $\iff$ any cone (g,h) over (f, f) has $g = h$ $\iff$ (g, h) factors uniquely through $(1_A, 1_A)$. $\square$

Hence, provided $\mathcal{D}$ has pullbacks, a morphism $\alpha : F \to G$ in $[\mathcal{C}, \mathcal{D}]$ is monic $\iff$ $\alpha_A : FA \to GA$ is monic for each A.

Theorem 4.8. *If $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in $\mathcal{D}$.*

Proof. Suppose $\mathcal{C}$ and $\mathcal{D}$ both have limits of shape J and let $(F \dashv G)$. The diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\triangle"] & \mathcal{D} \arrow[d, "\triangle"] \\ {[J, \mathcal{C}]} \arrow[r, "{[J,F]}"] & {[J, \mathcal{D}]} \end{tikzcd}$$

commutes and $[J, F]$ has a right adjoint $[J, G]$. So by 3.5 the diagram of right adjoints

$$\begin{tikzcd} {[J, \mathcal{D}]} \arrow[r, "{[J,G]}"] \arrow[d, "\lim_J"] & {[J, \mathcal{C}]} \arrow[d, "\lim_J"] \\ \mathcal{D} \arrow[r, "G"] & \mathcal{C} \end{tikzcd}$$

commutes up to isomorphism, i.e. G preserves limits of shape J. □

Proof. Let $D : J \to \mathcal{D}$ be a diagram with limit $(L, (\lambda_j \mid j \in \mathrm{ob}\, J))$. Given a cone $(A, (\mu_j : A \to GD(j) \mid j \in \mathrm{ob}\, J))$ in $\mathcal{C}$, we get a cone $(FA, (\bar{\mu_j} : FA \to D(j) \mid j \in \mathrm{ob}\, J))$ in $\mathcal{D}$, and hence a unique $\bar{\nu} : FA \to L$ such that $\lambda_j \bar{\nu} = \bar{\mu_j}$ for all j.

Then $\nu : A \to GL$ is the unique morphism such that $(G\lambda_j)\nu = \mu_j \forall j$. □

The 'primeval' Adjoint Functor Theorem says that if $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits, then G has a left adjoint.

This depends on two ideas:

Lemma 4.9. *$\mathcal{C}$ has an initial object $\iff$ $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ has a limit.*

Proof. Suppose $\mathcal{C}$ has an initial object 0. The morphisms $(0 \to A \mid A \in \mathrm{ob}\, \mathcal{C})$ form a cone over $1_{\mathcal{C}}$. If we had another, say $(L, (\lambda_A \mid A \in \mathrm{ob}\, \mathcal{C}))$, then $\lambda_0 : L \to 0$ would make

$$\begin{tikzcd} L \arrow[r, "\lambda_0"] \arrow[dr, "\lambda_A"'] & 0 \arrow[d] \\ & A \end{tikzcd}$$

commute for all A, and it's the only morphism which does.

Conversely, suppose $(I, (\lambda_A : I \to A \mid A \in \mathrm{ob}\, \mathcal{C}))$ is a limit for $1_{\mathcal{C}}$.

If $f : I \to A$, then

$$\begin{tikzcd} I \arrow[r, "\lambda_I"] \arrow[dr, "\lambda_A"'] & I \arrow[d, "f"] \\ & A \end{tikzcd}$$

commutes. In particular, $\lambda_A \lambda_I = \lambda_A$ for all A, so $\lambda_I = 1_I$ since both are factorisations of the limit cone through itself. So $f = \lambda_A$, and hence I is initial. □

Lemma 4.10. *Suppose $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then, for each $A \in \operatorname{ob}\mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $(A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Suppose given $D : J \to (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j : A \to GUD(j))$ for each j. Let $(L, (\lambda_j \mid j \in \operatorname{ob} J))$ be a limit for UD, then $(GL, (G\lambda_j \mid j \in \operatorname{ob} J))$ is a limit for GUD. But the f_j form a cone over GUD with apex A, so there's a unique $h : A \to GL$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f_j"'] & GL \arrow[d, "G\lambda_j"] \\ & GUD(j) \end{tikzcd}$$

commutes for all j. So there's a unique lifting of the cone over D in $(A \downarrow G)$.

Suppose we're given a cone $((B, g), (\mu_j \mid j \in \operatorname{ob} J))$ over D. Then

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "h"'] & GB \arrow[d, "G_k"] \\ & GL \end{tikzcd}$$

commutes since both ways round are factorisations of $(f_j \mid j \in \operatorname{ob} J)$ through the limit GL. □

Combining 4.10 and 4.9 with 3.2, we've proved the primeval Adjoint Functor Theorem. However, this requries $\mathcal{D}$ to have limits for diagrams 'as big as $\mathcal{D}$ itself', and the only such categories are preorders (c.f. Q6, sheet 2).

In practice, the most we can hope for is that $\mathcal{D}$ has all small limits. We call such a $\mathcal{D}$ **complete**.

Theorem 4.11 (General Adjoint Functor Theorem)**.** *Suppose that $\mathcal{D}$ is complete and locally small. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and satisfies the 'solution set condition': for any $A \in \operatorname{ob}\mathcal{C}$, there is a set $\{f_i : A \to GB_i \mid i \in I\}$ of objects of $(A \downarrow G)$ such that any $h : A \to GC$ factors as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} Gc$$

for some $i \in I$ and $g : B_i \to C$.

Proof. If G has a left adjoint, then it preserves small limits by 4.8, and $\{\eta_A : A \to GFA\}$ is a singleton solution set at A.

Conversely, each $(A \downarrow G)$ is complete by 4.10, and locally small since it admits a faithful functor to $\mathcal{D}$. So we need to show: if $\mathcal{A}$ is complete and locally small, and has a weakly initial set of objects $\{S_i \mid i \in I\}$, then $\mathcal{A}$ has an initial object.

First form $P = \prod_{i \in I} S_i$: then P is weakly initial.

Now form the limit $I \xrightarrow{a} P$ of the diagram $P \rightrightarrows P$ whose edges are all morphism $P \to P$ in $\mathcal{A}$.

Claim I is initial: it's weakly initial since it admits a morphism to P.

Suppose we had $I \overset{f}{\underset{g}{\rightrightarrows}} A$. Let $b : E \to I$ be an equaliser for (f, g): then there exists $c : P \to E$.

Now $P \xrightarrow{c} E \xrightarrow{b} I \xrightarrow{a} P$ is an edge of the diagram whose limit is I, but so is 1_P; so $abca = 1_P a = a$. But a is monic, so $bca = 1_I$. So b is (split) epic, and $f = g$. So all the $(A \downarrow G)$ have initial objects, hence by 3.2 G has a left adjoint. $\square$

The Special Adjoint Functor Theorem imposes additional conditions on $\mathcal{C}$ and $\mathcal{D}$ which ensure that every functor $\mathcal{D} \to \mathcal{C}$ preserving small limits has a left adjoint.

Definition 4.12. a. A **subobject** of an object A is a monomorphism $A' \rightarrowtail A$. We write $\mathbf{Sub}_{\mathcal{C}}(A)$ for the full subcategory of $\mathcal{C}/A$ whose objects are subobjects of A: note that this category is a preorder.

b. We say $\mathcal{C}$ is **well-powered** if each $\mathbf{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category, i.e. up to isomorphism each object has only a set of subobjects.

Dually, $\mathcal{C}$ is **well-copowered** if $\mathcal{C}^{op}$ is well-powered.

Lemma 4.13. *Suppose given a pullback*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"] & A \arrow[d, tail, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

with f monic. Then h is monic.

Proof. Suppose $D \overset{x}{\underset{y}{\rightrightarrows}} P$ satisfy $hx = hy$. Then $fkx = fky = ghx = ghy$ and f is monic so $kx = ky$.

Now $x = y$ since both are factorisations of the same cone through the pullback. $\square$

Theorem 4.14 (Special Adjoint Functor Theorem)**.** *Suppose both $\mathcal{C}$ and $\mathcal{D}$ are locally small, and $\mathcal{D}$ is complete, well-powered and has a separating set. Then $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint $\iff$ G preserves all small limits.*

Proof. The forward implication is 4.8 again.

Conversely, we first show that $(A \downarrow G)$ has the properties we've assumed for $\mathcal{D}$: it's complete by 4.10, and locally small as in 4.11. It's well-powered since subobjects of (B, f) in $(A \downarrow G)$ are in bijection with subobjects $B' \rightarrowtail B$ such that f factors through $GB' \rightarrowtail GB$.

It has a coseparating set: if $\{S_i \,|\, i \in I\}$ is a coseparating set for $\mathcal{D}$, then $\{(S_i, f) \,|\, i \in I, f : A \to GS_i\}$ is a coseperating set for $(A \downarrow G)$, since if $(B, f) \underset{g'}{\overset{g}{\rightrightarrows}} (B', f')$ satisfies $g \neq g'$, there exists $h : B' \to S_i$ for some i with $hg \neq hg'$, and then h is a morphism $(B', f') \to (S_i, (Gh)f')$ in $(A \downarrow G)$.

Now we show that if $\mathcal{A}$ is complete, locally small and well-powered and has a coseperating set, then it has an initial object.

First form $P = \prod_{i \in I} S_i$, where $\{S_i \,|\, i \in I\}$ is a coseparating set.

Consider the diagram

$$\begin{tikzcd}
P' \arrow[dr, tail] & \\
P'' \arrow[r, tail] & P \\
\vdots & \\
P^{(n)} \arrow[uur, tail] &
\end{tikzcd}$$

whose edges are a representative set of subobjects of P.

Form its limit

$$\begin{tikzcd}
 & P' \\
I \arrow[ur] \arrow[r] \arrow[ddr] & P'' \\
 & \vdots \\
 & P^{(n)}
\end{tikzcd}$$

by the argument of 4.13 the legs $I \to P^{(-)}$ are monic, so $I \rightarrowtail P$ is monic and it's the least subobject of P.

Hence in particular I has no proper subobjects, so any two maps $I \underset{g}{\overset{f}{\rightrightarrows}} A$ must be equal, since their equaliser is an isomorphism.

Now given $A \in \mathcal{A}$, form the product $Q = \prod_{i, f : A \to S_i} S_i$. The canonical morphism $h : A \to Q$ defined by $\pi_{i,f} h = f$ is monic since the S_i form a coseparating set.

We also have $k : P \to Q$ defined by $\pi_{i,f} k = \pi_i$, and we can form the pullback

$$\begin{tikzcd}
I \arrow[r] \arrow[dr, tail] & B \arrow[r, "m"] \arrow[d, tail, "l"] & A \arrow[d, tail, "h"] \\
 & P \arrow[r, "k"] & Q
\end{tikzcd}$$

By 4.13 l is monic and hence isomorphic to an edge of the diagram defining I, so $I \rightarrowtail P$ factors through it. So there exists a morphism $I \to A$, hence I is initial. $\square$

5 Monads

Suppose given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $F \dashv G$. How much of this can we describe purely in terms of $\mathcal{C}$?

We have the composite $T = GF : \mathcal{C} \to \mathcal{C}$, and the unit $\eta : 1_{\mathcal{C}} \to T$. We also have $G\epsilon_F : GFGF \to GF$, which we'll denote $\mu : TT \to T$.

These satisfy the commutative diagrams

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] \arrow[dl, phantom, "1"] \arrow[dr, phantom, "2"] & T \arrow[l, "\eta_T"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] \arrow[dr, phantom, "3"] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd}$$

from the $\triangle^r$ identities and naturality of ϵ.

Definition 5.1. A **monad** $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \to T$, $\mu : TT \to T$ satisfying the commutative diagrams 1, 2 and 3.

Definition 5.2. Let $\mathbb{T}$ be a monad on $\mathcal{C}$. A **$\mathbb{T}$-algebra** is a pair (A, α) where $A \in \operatorname{ob}\mathcal{C}$, and $\alpha : TA \to A$ satisfies

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \arrow[dl, phantom, "4"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "5"] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A \end{tikzcd}$$

A **homomorphism** $f : (A, \alpha) \to (B, \beta)$ of $\mathbb{T}$-algebras is a morphism $f : A \to B$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "6"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of $\mathbb{T}$-algebras.

Lemma 5.3. *The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ has a left adjoint F, and the adjunction $(F \dashv G)$ induces the monad $\mathbb{T}$.*

Proof. We define $FA = (TA, \mu_A)$ (which is an algebra by ② and ③), and $F(A \xrightarrow{f} B) = Tf$ (which is a homomorphism by naturality of μ).

Clearly $GF = T$ and $\eta : 1_{\mathcal{C}} \to GF$.

We define $\epsilon : FG \to 1_{\mathcal{C}^{\mathbb{T}}}$ by $\epsilon_{(A,\alpha)} = \alpha : (TA, \mu_A) \to (A, \alpha)$ (which is a homomorphism by ⑤).

The triangular identities for η and ϵ follow from ④ and ①, so $(F \dashv G)$.

Finally, $G_{\epsilon_{FA}} = \mu_A$ by the definitions of FA and ϵ, so the adjunction incudes $\mathbb{T}$. $\square$

Note that if $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ induces $\mathbb{T}$, then so does $\mathcal{C} \underset{G/\mathcal{D}'}{\overset{F}{\rightleftarrows}} \mathcal{D}'$ where $\mathcal{D}'$ is the full subcategory of objects of the form FA. So in seeking to construct $\mathcal{D}$, we may require F to be bijective on objects. But then morphisms $FA \to FB$ in $\mathcal{D}$ correspond bijectively to morphisms $A \to GFB = TB$ in $\mathcal{C}$.

Definition 5.4. Given a monad $\mathbb{T}$ on $\mathcal{C}$, the **Kleisi category** $\mathcal{C}_{\mathbb{T}}$ is defined by: $\mathrm{ob}\, \mathcal{C}_{\mathbb{T}} = \mathrm{ob}\, \mathcal{C}$, morphisms $A \to B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $A \to TB$ in $\mathcal{C}$, the identity $A \to A$ is $A \xrightarrow{\eta_A} TA$, and the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} C$.

We check

$$\begin{aligned}
A \xrightarrow{1_A} A \xrightarrow{f} B &= A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} B \\
&= A \xrightarrow{f} TA \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} B \\
&= f \text{ by ②}
\end{aligned}$$

$$\begin{aligned}
A \xrightarrow{f} B \xrightarrow{1_B} B &= A \xrightarrow{f} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} B \\
&= f \text{ by ①}
\end{aligned}$$

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$$\begin{aligned}
(hg)f &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \text{ by ③} \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{Th} TTD \xrightarrow{\mu_D} TD \\
&= h(gf)
\end{aligned}$$

Lemma 5.5. *There exists an adjunction* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_{\mathbb{T}}$ *inducing* $\mathbb{T}$.

Proof. We define $FA = A$ and $F(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. This clearly preserves identities, and

$$\begin{aligned}(Fg)(Ff) &= A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC \\ &= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \text{ by (1) and naturality of } \eta \\ &= F(gf)\end{aligned}$$

We define $GA = TA$ and $G(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. G preserves identities by (1) and

$$\begin{aligned}G(A \xrightarrow{f} B \xrightarrow{g} C) &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \text{ by (3)} \\ &= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \text{ by naturality of } \mu \\ &= (Gg)(Gf)\end{aligned}$$

Clearly $GFA = TA$ and

$$\begin{aligned}GF(A \xrightarrow{f} B) &= TA \xrightarrow{Tf} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB \\ &= Tf \text{ by (1)}\end{aligned}$$

so $GF = T$ and $\eta : 1_{\mathcal{C}} \to GF$.

We define $FGA \xrightarrow{\epsilon_A} A$ to be $TA \xrightarrow{\eta_{TA}} TA$. To verify naturality of ϵ, consider

$$\begin{tikzcd} FGA \arrow[r, "FGf"] \arrow[d, "\epsilon_A"] & FGB \arrow[d, "\epsilon_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The top and right edges yield

$$TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{1_{TTB}} TTB \xrightarrow{\mu_B} TB$$

and the left and bottom yield

$$TA \xrightarrow{1_{TA}} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$$

For the $\triangle^r$ identities,

$$GA \xrightarrow{\eta_{GA}} GFGA \xrightarrow{G\epsilon_A} GA = TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = 1_{TA}$$

and

$$\begin{aligned}FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA &= A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA \\ &= A \xrightarrow{\eta_A} TA \; (= FA \xrightarrow{1_{FA}} FA)\end{aligned}$$

Finally, $G\epsilon_{FA} = TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = \mu_A$, so the adjunction induces the monad $\mathbb{T}$. $\square$

Theorem 5.6. *Given a monad $\mathbb{T}$ on $\mathcal{C}$, let* $\boldsymbol{Adj}(\mathbb{T})$ *be the category whose objects are adjunctions* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *inducing* $\mathbb{T}$*, and whose morphisms* $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}) \to (\mathcal{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{D}')$ *are functors* $K : \mathcal{D} \to \mathcal{D}'$ *satisfying* $KF = F'$ *and* $G'K = G$*.*

Then the Kleisi category $\mathcal{C}_{\mathbb{T}}$ *is initial in* $\boldsymbol{Adj}(\mathbb{T})$*, and the Eilenberg-Moore category* $\mathcal{C}^{\mathbb{T}}$ *is terminal.*

Proof. Given $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$ in $\mathbf{Adj}(\mathbb{T})$, we define the **Eilenberg-Moore comparison functor** $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\epsilon_B)$ (note that $G\epsilon_B$ is an algebra structure on GB: the unit condition ④ follows from a $\triangle^r$ identity, and ⑤ follows from the naturality of ϵ).

$K(B \xrightarrow{g} B') = Gg : (GB, G\epsilon_B) \to (GB', G\epsilon_{B'})$ (a homomorphism since ϵ is natural).

It's clear that K is a functor, that $G^{\mathbb{T}}K = G$ and that $KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ and $KF(A \xrightarrow{f} B) = Tf = F^{\mathbb{T}}f$.

Uniqueness: suppose $\bar{K}$ also satisfies $G^{\mathbb{T}}\bar{K} = G$ and $\bar{K}F = F^{\mathbb{T}}$. Then $\bar{K}B$ is of the form (GB, β_B) for some algebra structure β_B, and that $\beta_{FA} = \mu_A = G\epsilon_{FA}$ for all A.

Given any B, consider the diagram

$$\begin{tikzcd} GFGFGB \arrow[r, "GFG\epsilon_B"] \arrow[d, "\mu_{GB}"] & GFGB \arrow[d, "\beta_B"] \\ GFGB \arrow[r, "G\epsilon_B"] & GB \end{tikzcd}$$

which must commute, since $G\epsilon_B$ is an algebra homomorphism. But it would also commute with $G\epsilon_B$ in place of β_B, and $GFG\epsilon_B$ is (split) epic, so $\beta_B = G\epsilon_B$.

For the **Kleisi comparison functor** $K : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$, we define $KA = FA$, $K(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB$.

To verify this is functorial, consider

$$\begin{aligned} K(A \xrightarrow{f} B \xrightarrow{g} C) &= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{FG\epsilon_{FC}} FGFC \xrightarrow{\epsilon_{FC}} FC \\ &= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{\epsilon_{FGFC}} FGFC \xrightarrow{\epsilon_{FC}} FC \\ &= FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB \xrightarrow{Fg} FGFC \xrightarrow{\epsilon_{FC}} FC \\ &= (Kg)(Kf) \end{aligned}$$

$GKA = GFA = TA = G_{\mathbb{T}}A$
$GK(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB = G_{\mathbb{T}}(f)$
And $KF_{\mathbb{T}}A = FA$,

$$KF_{\mathbb{T}}(A \xrightarrow{f} B) = \begin{tikzcd} FA \arrow[r, "Ff"] & FB \arrow[r, "F\eta_B"] \arrow[dr, "1_{FB}"] & FGFB \arrow[d, "\epsilon_{FB}"] \\ & & FB \end{tikzcd}$$

So K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Uniqueness: suppose $\bar{K}$ is any other morphism $\mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ in $\mathbf{Adj}(\mathbb{T})$. Then $\bar{K}A = FA = KA$ for all A; since $\bar{K}$ commutes with both the Fs and the Gs, we have $\bar{K}(\epsilon_A) = \epsilon_{FA}$.

We can write $A \xrightarrow{f} B$ as $A \xrightarrow{F_{\mathbb{T}} f} F_{\mathbb{T}} G_{\mathbb{T}} \xrightarrow{\epsilon_B} B$, so $\bar{K}(f) = \bar{K}(\epsilon_B) F f = K(f)$. $\square$

The Kleisi category $\mathcal{C}_{\mathbb{T}}$ inherits coproducts from $\mathcal{C}$ if $\mathcal{C}$ has them, but it has few other limits or colimits in general.

Theorem 5.7. *i. The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates all limits which exist in $\mathcal{C}$.*

ii. If $T : \mathcal{C} \to \mathcal{C}$ preserves colimits of shape J, then $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates them.

Proof. i. Let $D : J \to \mathcal{C}^{\mathbb{T}}$ be a diagram, write $D(j) = (GD(j), \delta_j)$.

Let $(L, (\lambda_j : L \to GD(j)))$ be a limit for GD. The composites $TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over GD, since the edges of GD are algebra homomorphisms.

So they induce a unique $l : TL \to L$ such taht

$$\begin{tikzcd} TL \arrow[r, "T\lambda_j"] \arrow[d, "l"] & TGD(j) \arrow[d, "\delta_j"] \\ L \arrow[r, "\lambda_j"] & GD(j) \end{tikzcd}$$

commutes for each j.

l is an algebra structure: $l\eta_L = l_L$ since both are factorisations of (λ_j) through itself, and $lTl = l\mu_L$ since they're factorisations of the same cone through L.

So $((L, l), (\lambda_j))$ is the unique lifting of $(L, (\lambda_j))$ to a cone over D in $\mathcal{C}^{\mathbb{T}}$.

Any cone over D in $\mathcal{C}^{\mathbb{T}}$ factors uniquely through L, and the factorisation is an algebra homomorphism.

ii. Similarly, given $D : J \to \mathcal{C}^{\mathbb{T}}$ as before and a colimit $(L, (\lambda_j : GD(j) \to L))$ for GD, we get a unique $l : TL \to L$ making

$$\begin{tikzcd} TGD(j) \arrow[r, "T\lambda_j"] \arrow[d, "\delta_j"] & TL \arrow[d, "l"] \\ GD(j) \arrow[r, "\lambda_j"] & L \end{tikzcd}$$

commute, since $(TL, (T\lambda_j))$ is a colimit. The rest of the proof is similar to (i).

$\square$

Definition 5.8. An adjunction $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$, $(F \dashv G)$, is **monadic** if the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is part of an equivalence, where $\mathbb{T}$ is the monad induced by $(F \dashv G)$. We also say $G : \mathcal{D} \to \mathcal{C}$ is monadic if it has a left adjoint and the adjunction is monadic.

Note that K preserves all limits which exist in $\mathcal{D}$, since G preserves them and $G^{\mathbb{T}}$ creates them.

Lemma 5.9. *Suppose given* $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$, $(F \dashv G)$ *inducing a monad* $\mathbb{T}$ *on* $\mathcal{C}$. *Suppose, for each* $\mathbb{T}$*-algebra* (A, α), *the pair* $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ *has a coequaliser in* $\mathcal{D}$. *Then* $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ *has a left adjoint* L.

Proof. We define $L(A, \alpha) = \mathrm{coeq}(FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA)$.

Given $(A, \alpha) \xrightarrow{f} (B, \beta)$, we get

$$\begin{tikzcd}
FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] \arrow[d, "FGFf"] & FA \arrow[r] \arrow[d, "Ff"] & L(A,\alpha) \arrow[d, dashed, "Lf"] \\
FGFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\epsilon_{FB}"'] & FB \arrow[r] & L(B,\beta)
\end{tikzcd}$$

So $\exists! Lf$ making the right hand square commute. Uniqueness ensures L is functorial.

Morphisms $L(A, \alpha) \to B$ in $\mathcal{D}$ correspond bijectively to morphisms $f : FA \to B$ such that $f(F\alpha) = f(\epsilon_{FA})$ and hence to morphisms $\bar{f} : A \to GB$ such that $f\bar{\alpha} = Gf|_{GFA} = G\epsilon_B \circ GF\bar{f}$, i.e. to algebra homomorphisms $(A, \alpha) \to (GB, G\epsilon_B) = KB$.

So $(L \dashv K)$.

$\square$

Definition 5.10. a. A parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is **reflexive** if $\exists r : B \to A$ such that $fr = gr = 1_B$.

Note that $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ is reflexive, with common splitting $FA \xrightarrow{F\eta_A} FGFA$.

A **reflexive coequaliser** is the coequaliser of a reflexive pair.

b. A **split coequaliser diagram** is a diagram

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=60, "t"] \arrow[r, "h"] & C \arrow[l, bend left=40, "s"]
\end{tikzcd}
$$

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.

If these equations hold, h **is** a coequaliser of f and g: given $k : B \to D$ with $kf = kg$, we have $k = kgt = kft = ksh$, so k factors through h, uniquely since h is split epic.

Note that **any** functor preserves split equalisers.

c. Given $G : \mathcal{D} \to \mathcal{C}$, a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{D}$ is **G-split** if there exists a split coequaliser

$$
\begin{tikzcd}
GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[l, bend left=60, "t"] \arrow[r] & C \arrow[l, bend left=40, "s"]
\end{tikzcd}
$$

in $\mathcal{C}$.

The pair $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ of 5.9 is G-split:

$$
\begin{tikzcd}
GFGFA \arrow[r, shift left, "Gf\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & GFA \arrow[l, bend left=60, "\eta_{GFA}"] \arrow[r, "\alpha"] & A \arrow[l, bend left=40, "\eta_A"]
\end{tikzcd}
$$

is a split coequaliser diagram.

Theorem 5.11 (Precise Monadicity Theorem)**.** *$G : \mathcal{D} \to \mathcal{C}$ is monadic $\iff$ G has a left adjoint, and c reates coequalisers of G-split pairs.*

Theorem 5.12 (Crude Monadicity Theorem)**.** *Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, that $\mathcal{D}$ has and G preserves reflexive coequalisers, and G reflects isomorphisms. Then G is monadic.*

Proof. For the forward implication in 5.11, it's enough to show that $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates coequalisers of $G^{\mathbb{T}}$-split pairs. This follows from 5.7, given that T and TT both preserve split coequalisers.

Conversely in either case, $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L by 5.9. Now $LKB = \text{coeq}(FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB)$ and the counit $LKB \to B$ is the factorisation of $FGB \underset{\epsilon_B}{\overset{\epsilon_{FGB}}{\rightrightarrows}} B$ through this coequaliser.

But
$$
\begin{tikzcd}
GFGFGB \arrow[r, shift left] \arrow[r, shift right] & GFGB \arrow[l, bend left=60, "\eta_{GFGB}"] \arrow[r] & GB \arrow[l, bend left=40, "\eta_{GB}"]
\end{tikzcd}
$$
is a split coequaliser diagram.

So either set of hypotheses ensures that $LKB \to B$ is an isomorphism.

$KL(A, \alpha) = K(\text{coeq}(FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA))$. Either hypothesis implies that $G = G^{\mathbb{T}}K$ preserves this coequaliser, but

$$\begin{tikzcd}
GFGFGA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[l, bend left=50, "\eta_{GFA}"] & A \arrow[l, bend left, "\eta_A"]
\end{tikzcd}$$

is a split coequaliser, so $GL(A, \alpha) \cong A$.

The unit $(A, \alpha) \to KL(A, \alpha)$ is mapped to this isomorphism by $G^{\mathbb{T}}$, so it's an isomorphism in $\mathcal{C}^{\mathbb{T}}$. $\square$

Remark 5.13. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction, and suppose $\mathcal{D}$ has reflexive coequalisers. The **monadic tower** of $(F \dashv G)$ is the diagram

$$\begin{tikzcd}
 & & (\mathcal{C}^{\mathbb{T}})^{\mathbb{S}} \arrow[dl, shift left] \arrow[dll, shift left] & \cdots \\
\mathcal{D} \arrow[urr, shift left] \arrow[r, shift left, "K"] \arrow[d, shift left, "G"] & \mathcal{C}^{\mathbb{T}} \arrow[l, shift left, "L"] \arrow[ur, shift left] \arrow[dl, shift left, "G^{\mathbb{T}}"] & & \\
\mathcal{C} \arrow[u, shift left, "F"] \arrow[ur, shift left, "F^{\mathbb{T}}"] & & &
\end{tikzcd}$$

where $\mathbb{T}$ is the monad induced by $(F \dashv T)$, K is the Eilenberg-Moore comparison functor, $(L \dashv K)$ (5.9), $\mathbb{S}$ is the monad induced by $(L \dashv K)$, etc.

We say $(F \dashv G)$ has **monadic length** n if we reach an equivalence after n steps. **Top** $\to$ **Set** has monadic length ∞.

6 Regular Categories

Definition 6.1. The **image** of a morphism $A \xrightarrow{f} B$ is the smallest subobject of B through which f factors, if this exists.

We say $\mathcal{C}$ **has images** if every $f \in \operatorname{mor} \mathcal{C}$ has an image.

$A \xrightarrow{f} B$ is a **cover** if its image is 1_B, i.e. it doesn't factor through any proper subobject of B. We write $A \overset{f}{\twoheadrightarrow} B$ to indicate that f is a cover.

Lemma 6.2. *If $\mathcal{C}$ has finite limits, then covers in $\mathcal{C}$ coincide with strong epimorphisms.*

Proof. Recall that f is strong epic if and only if given $(*)$ $\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & C \arrow[d, tail, "k"] \\ B \arrow[r, "h"] \arrow[ur, dashed, "l"] & D \end{tikzcd}$ with k monic, $\exists l : B \to C$ with $kl = h$ and $lf = g$.

Being a cover is the special case of this condition with $h = 1_B$, so strong epimorphisms are covers.

Conversely, if f is a cover then it's epic, since if $gf = hf$ then f factors through the equaliser of g and h, so this must be an isomorphism. Given $(*)$, we can form the pullback $\begin{tikzcd} P \arrow[r, "n"] \arrow[d, "m"] & C \arrow[d, tail, "k"] \\ B \arrow[r, "h"] & D \end{tikzcd}$, and m is monic by 4.13.

f factors through m, so m is an isomorphism and $B \overset{nm^{-1}}{\to} C$ is the diagonal fill in for $(*)$. $\square$

It follows that image factorisation is functorial: given

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, twoheadrightarrow] \arrow[ddd, "g"'] & & B \arrow[ddd, "h"] \\ & I \arrow[ur, tail] \arrow[d, dashed] & \\ & I' \arrow[dr, tail] & \\ A' \arrow[ur, twoheadrightarrow] \arrow[rr, "f'"'] & & B' \end{tikzcd}$$

we get a unique $I \to I'$ making everything commute. So image factorisation defines a functor $[\mathbf{2}, \mathcal{C}] \to [\mathbf{3}, \mathcal{C}]$.

Definition 6.3. A **regular category** is a category which preserves finite limits and images, in which strong epimorphisms are stable under pullback.

A **regular functor** is one which preserves finite limits and strong epimorphisms.

Theorem 6.4. *In a regular category, the strong epimorphisms coincide with regular epimorphisms.*

Proof. Regular $\implies$ strong is true in general (see sheet 1). Suppose $A \overset{f}{\twoheadrightarrow} B$ is strong epic. Let $R \underset{b}{\overset{a}{\rightrightarrows}} A$ be the **kernel-pair** of f, i.e. the pullback of f against itself. Certainly $fa = fb$.

Suppose $A \overset{g}{\to} C$ with $ga = gb$. Form the image

$$A \Longrightarrow I \overset{(k,l)}{\rightarrowtail} B \times C$$

of (f, g).

Claim k is an isomorphism: given this, the composite $B \overset{k^{-1}}{\to} I \overset{l}{\to} C$ satisfies $lk^{-1}f = lk^{-1}kh = lh = g$, and it's unique since f is epic.

We know k is strong epic, since $kh = f$ is strong epic. So we need to show it's monic.

Suppose $D \underset{y}{\overset{x}{\rightrightarrows}}$ satisfy $kx = ky$. Form the pullback

$$\begin{array}{ccccc} F & & \xrightarrow{m} & & D \\ \downarrow{\scriptstyle (n,p)} & & & & \downarrow{\scriptstyle (x,y)} \\ A \times A & \xrightarrow{1\times h} & A \times I & \xrightarrow{h\times 1} & I \times I \end{array}$$

then $h \times 1$ and $1 \times h$ are strong epimorphisms, and hence so is m.

Now $fn = khn = kxm = kym = khp = fp$, so (n,p) factors through (a,b), say by $E \xrightarrow{q} R$. Since $kha = khb$ and $lha = lhb$, and (k,l) is monic, we have $ha = hb$. So $xm = hn = haq = hbq = hp = ym$. But m is epic, so $x = y$. $\square$

Remark 6.5. In many textbooks, regular categories are defined as categories with (some) finite limits, in which every morphism factors as a regular epimorphism and a monomorphism, and regular epimorphims are stable under pullback.

Note that if $A \xrightarrow{f} B$ has kernel pair $R \overset{a}{\underset{b}{\rightrightarrows}} A$ and image factorisation $A \overset{g}{\twoheadrightarrow} I \rightarrowtail B$, then (a,b) is also the kernel-pair of g since h is monic. So g may be obtained as the coequaliser of the kernel-pair of f, and f as the factorisation of f through this.

Definition 6.6. Let $R \overset{a}{\underset{b}{\rightrightarrows}} A$ be a parallel pair in a category with finite limits.

a. (a,b) is a **relation** if $R \xrightarrow{(a,b)} A \times A$ is monic.

b. (a,b) is **reflexive** if there exists $A \xrightarrow{r} R$ with $ar = br = 1_A$.

c. (a,b) is **symmetric** if there exists $R \xrightarrow{s} R$ with $as = b$, $bs = a$.

d. (a,b) is **transitive** if, given the pullback $\begin{array}{ccc} T & \xrightarrow{q} & R \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle a} \\ R & \xrightarrow{b} & A \end{array}$, there exists $t : T \to R$ such that $at = ap$ and $bt = bq$.

e. (a,b) is an **equivalence relation** if all of (a-d) hold.

The kernel-pair of any $A \xrightarrow{f} B$ is an equivalence relation.

(a,b) is an **effective** equivalence relation if it occurs as a kernel pair, $\mathcal{C}$ is an **effective** regular category if all equivalence relations in $\mathcal{C}$ are effective (aka 'Barr-exact').

Definition 6.7. The **support** $\sigma(A)$ of an object A in a regular category is the image of $A \to 1$. A is **well-supported** if $\sigma(A) \cong 1$, i.e. if $A \twoheadrightarrow 1$ is strong epic.

$\mathcal{C}$ is **totally supported** if all its objects are well-supported. (E.g. **Gp** and **ApGp** are totally supported, since any $A \to 1$ is split epic).

An object 0 in a regular category is **strict** if any morphism $A \to 0$ is an isomorphism. (This implies that 0 is initial: $O \times A \xrightarrow{\pi_1} 0$ is an isomorphism, so $0 \xrightarrow{\pi_1^{-1}} 0 \times A \xrightarrow{\pi_2} A$ exists for any A, but given $O \overset{f}{\underset{g}{\rightrightarrows}} A$, the equaliser must be an isomorphism.)

$\mathcal{C}$ is **almost totally supported** if every object of $\mathcal{C}$ is either well-supported or strict (e.g. **Set**).

Theorem 6.8 (Barr's Embedding Theorem)**.** *Let $\mathcal{C}$ be a small regular category. Then there exists a small category $\mathcal{D}$ and a full and faithful regular functor $\mathcal{C} \to [\mathcal{D}, \boldsymbol{Set}]$. Moreover, if $\mathcal{C}$ is almost totally supported then $\mathcal{D}$ can be taken to be a monoid.*

We'll prove the most important part of this: $\mathcal{C}$ has an isomorphism reflecting regular functor to be a power of **Set**. We follow a proof due to F. Borceaux:

Theorem 6.9. *Let $\mathcal{C}$ be a small a.t.s. regular category. Then there exists an isomorphism reflecting regular functor $F : \mathcal{C} \to \boldsymbol{Set}$.*

Proof. We construct F as the colimit in $[\mathcal{C}, \mathbf{Set}]$ of a diagram of representable functors: explicitly, J will be a meet-semilattice and $D : J \to \mathcal{C}$ a diagram such that each $D(j)$ is well-supported and each $D(j' \to j)$ is a strong epimorphism $D(j') \twoheadrightarrow D(j)$.

Then $F = colim(J^op \xrightarrow{D} \mathcal{C}^op \xrightarrow{Y} [\mathcal{C}, \mathbf{Set}])$.

Explicitly, elements of FA are represented by morphisms $D(j) \xrightarrow{f} A$ for some j, where $f \sim f'$ if and only if

$$\begin{tikzcd} & D(j) \arrow[dr, "f"] & \\ D(j \wedge j') \arrow[ur] \arrow[dr] & & A \\ & D(j') \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes.

F preserves finite products: $F1 = \{*\}$, and if $D(j) \xrightarrow{f} A$, $D(j') \xrightarrow{g} B$ represent elts of FA and FB, $D(j \wedge j') \to D(j) \xrightarrow{f} A$ and $D(j \wedge j') \to D(j) \xrightarrow{g} B$ induce an element of $F(A \times B)$, mapping to the given element of $FA \times FB$.

Hence $F(A \times B) \to FA \times FB$ is surjective, and it's easily seen to be injective.

F preserves equalisers: notes that if 0 exists in $\mathcal{C}$, then $F0 = \emptyset$. If

$$E \overset{e}{\rightarrowtail} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

is an equaliser diagram in $\mathcal{C}$ and E is well-supported, then the equaliser of $FA \rightrightarrows FB$ consists of morphisms $D(j) \to A$ having equal composites with f

and g (and hence factoring through E). And if $E = 0$ then the equaliser of $FA \rightrightarrows FB$ is $\emptyset$.

Now assume that, for every strong epimorphism $A \xrightarrow{f} D(j)$ in $\mathcal{C}$, there exists $j' \leq j$ such that $D(j' \to j) = f$.

Then F preserves strong epimorphisms: given $A \overset{f}{\twoheadrightarrow} B$ and a morphism $D(j) \xrightarrow{g}$ representing an element of FB, form the pullback

$$\begin{tikzcd} D(j') \arrow[r, "h"] \arrow[d] & A \arrow[d, Rightarrow, "f"] \\ D(j) \arrow[r, "g"] & B \end{tikzcd}$$

then h represents an element of FA whose image under Ff is g. So Ff is surjective.

Assume every well-supported object of $\mathcal{C}$ occurs as $D(j)$ for some j. Then F preserves properness of subobjects: the element of FA represented by $D(j) \xrightarrow{1} A$ can't be in the image of $FA' \rightarrowtail FA$ for any proper subobject $A' \rightarrowtail A$ (if it were, we'd have $\begin{tikzcd} D(j') \arrow[r] \arrow[d, Rightarrow] & A' \arrow[d, tail] \\ D(j) \arrow[r, "1"] \arrow[ur, dashed] & A \end{tikzcd}$)

Since F preserves equalisers, it follows that it's faithful. Hence F reflects monomorphisms, and so the argument above shows that it reflects isomorphisms.

We'll construct J as the union $\bigcup_{n=0}^{\infty} J_n$ of an increasing sequence of sub-semilattices J_n: $J_0 = \{1\}$ and $D(1) = 1$, the terminal object of $\mathcal{C}$. Objects of $J_1 \backslash J_0$ are non-empty finite sets $\{A_1, A_2, \ldots, A_n\}$ of well-supported objects of $\mathcal{C}$, ordered by $\supseteq$ (so $j \wedge j' = j \cup j'$), and $D(\{A_1, A_2, \ldots, A_n\}) = \prod_{i=1}^{n} A_i$. (Note that is well-supported: if A and B are well supported, we have a pullback $\begin{tikzcd} A \times B \arrow[r, Rightarrow] \arrow[d, Rightarrow] & A \arrow[d, Rightarrow] \\ B \arrow[r, Rightarrow] & 1 \end{tikzcd}$)

If $j' \supseteq j$, $D(j \to j')$ is the product projection $\prod_{A \in j'} \twoheadrightarrow \prod_{A \in j} A$.

An object of $J_2 \backslash J_1$ is a pair $(j_1, \{f_1, \ldots, f_n\})$ where $j_1 \in J_1 \backslash J_0$ and $\{f_1, \ldots, f_n\}$ is a non-empty finite set of strong epimorphisms with codomain $D(j_1)$. (Equivalently, well-supported objects of $\mathcal{C}/D(j_1)$.

The meet of $(j_1, \{f_1, \ldots, f_n\})$ and $(j_1', \{g_1, \ldots g_m\})$ has first coordinate $j_1 \wedge j_1'$, and then we take the union of the sets of strong epimorphisms obtained from pulling back the f_i and g_j along $D(j_1) \wedge j_1' \twoheadrightarrow D(j_1)$ and $D(j_1 \wedge j_1') \twoheadrightarrow D(j_1')$.

We define $D((j_1, \{f_1, \ldots, f_n\}))$ to be the domain of the object $\prod_{i=1}^{n} f_i$ of $\mathcal{C}/D(j_1)$, and $D(j_2 \wedge j_2')$ is the composite of the appropriate product projection in $\mathcal{C}/D(j_1 \wedge j_1')$ with the appropriate pullback of $D(j_1 \wedge j_1') \twoheadrightarrow D(j_1)$.

Similarly, object of $J_3 \backslash J_2$ are pairs $(j_2, \{h_1, \ldots, h_n\})$ where $j_2 \in J_2 \backslash J_1$ and the h_i are strong epimorphisms with codomain $D(j_2)$, and so on.

Now we've satisfied the condition for F to preserve strong epimorphisms: if $A = D(j)$ where $j \in J_n \backslash J_{n-1}$, and $B \overset{g}{\twoheadrightarrow} A$ is strong epic, then $B = D((j, \{g\}))$ and $g = D((j, \{g\}) \to j)$. $\square$

Remark 6.10. If we define M to be the monoid of endomorphisms of $F : \mathcal{C} \to \mathbf{Set}$ in $[\mathcal{C}, \mathbf{Set}]$, then M acts on every FA, so we can regard F as taking values in $[M, \mathbf{Set}]$. As such, it's still regular and faithful, but also full.

Given a general regular category $\mathcal{C}$ and $S \rightarrowtail 1$ in $\mathcal{C}$, we write $\mathcal{C}_S$ for the full subcategory of $\mathcal{C}$ on objects A with $\sigma(A) \cong S$. This is closed in $\mathcal{C}$ under non-empty finite products, images and pullbacks of strong epimorphisms (if we're given
$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, Rightarrow, "k"] & A \arrow[d, Rightarrow, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$
with f strong epic then k is strong epic, so $\sigma(P) = \sigma(B)$.) It doesn't have all finite limits, but if $D : J \to \mathcal{C}_S$ is a finite diagram whose limit in $\mathcal{C}$ is not in $\mathcal{C}_S$, then there are no cones over D in $\mathcal{C}_S$.

Definition 6.11. Given a category $\mathcal{C}$, let $\mathcal{C}^+$ denote the category whose objects are those of $\mathcal{C}$ plus a new object 0, with one morphism $0 \to A$ and no morphisms $A \to 0$ for each $A \in \operatorname{ob}\mathcal{C}$.

In $\mathcal{C}_S^+$, every finite diagram has a limit: if it lies in $\mathcal{C}_S$ and has a limit there, that is its limit in $\mathcal{C}_S^+$, otherwise its limit is the unique cone with apex 0.

$\mathcal{C}_S^+$ is regular: the new morphisms $0 \to A$ are monic, and hence their own images, and strong epimorphisms are still stable under pullback.

It's almost totally supported: note that $\mathcal{C}/S$ may be identified with the full subcategory of $\mathcal{C}$ on objects with support $\leq S$, so its well-supported objects are those of $\mathcal{C}_S$.

We have a functor $E : \mathcal{C}/S \to \mathcal{C}_S^+$ sending all objects of $\mathcal{C}_S$ to themselves, and everything else to 0. E is regular! And we have a regular functor $(-) \times S : \mathcal{C} \to \mathcal{C}/S$ (it preserves finite limits because it's right adjoint to the forgetful functor, and images because they're stable under pullback along $S \to 1$).

Theorem 6.12. *For every small regular category $\mathcal{C}$, there exists a set I and an isomorphism reflecting regular functor $\mathcal{C} \to \mathbf{Set}^I$.*

Proof. Take $I = \mathbf{Sub}_{\mathcal{C}}(1)$, and for each $S \in \mathcal{I}$ consider the composite

$$G_S \overset{(-)\times S}{\longrightarrow} \mathcal{C}/S \overset{E}{\to} \mathcal{C}_S^+ \overset{F_s}{\longrightarrow} \mathbf{Set}$$

where F_S is defined as in 6.9. The G_S are all regular, and they jointly reflect isomorphisms: if $A \overset{f}{\to} B$ is not an isomorphism in $\mathcal{C}$, let $S = \sigma(B)$. Then $E(f)$

is either f itself (if $\sigma(A) = S$) or $0 \to B$ (otherwise). In either case it's not an isomorphism, so its image under F_S is not an isomorphism. $\square$

7 Additive and Abelian Categories

Definition 7.1. Let $\mathcal{A}$ be a category equipped with a forgetful functor $U : \mathcal{A} \to \mathbf{Set}$. A locally small category $\mathcal{C}$ is **enriched** over $\mathcal{A}$ if

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$$

factors through U.

If $\mathcal{A} = \mathbf{Set}_*$, $\mathcal{C}$ is called a **pointed category** (i.e. we have a distingushed morphism $A \xrightarrow{0} B$) for each pair (A, B) satisfying $0f = 0$ and $g0 = 0$).

If $\mathcal{A} = \mathbf{CMon}$, $\mathcal{C}$ is called **semi-additive** (i.e. we have 0 and $(f, g) \mapsto f + g$ satisfying $(f + g)h = fh + gh$ and $k(f + g) = kf + kg$).

If $\mathcal{A} = \mathbf{AbGp}$, $\mathcal{C}$ is called **additive**.

We also talk about **pointed** and **(semi)-additive** functors between such categories.

Lemma 7.2. *a. For an object 0 of a pointed category, TFAE:*

i. 0 is terminal

ii. 0 is initial

iii. $1_0 = 0$

b. For three objects A, B, C of a semi-additive category, TFAE:

i. There exist $A \xleftarrow{\pi_1} C \xrightarrow{\pi_2} B$ making C a product of A and B

ii. There exist $A \xrightarrow{\nu_1} C \xleftarrow{\nu_2} B$ making C a coproduct of A and B

iii. There exist $A \underset{\pi_1}{\overset{\nu_1}{\rightleftarrows}} C \underset{\pi_2}{\overset{\nu_2}{\leftrightarrows}} B$ satisfying $\pi_1\nu_1 = 1_A$, $\pi_2\nu_2 = 1_B$, $\pi_1\nu_1 = 0 = \pi_2\nu_1$ and $\nu_1\pi_1 + \nu_2\pi_2 = 1_{\mathcal{C}}$

Proof. In each case we prove (i) $\iff$ (iii); (ii) $\iff$ (iii) is dual.

a. (i) $\implies$ (iii) since if 0 is terminal there's only one morphism $0 \to 0$.

   (iii) $\implies$ (i): if (iii) holds then any $f : A \to 0$ satisfies $f = 1_0 f = 0f = 0$.

b. (i) $\implies$ (iii): given π_1 and π_2 making C a product, we define ν_1 and ν_2 to be the unique morphisms satisfying the first four equations of (iii).

Then

$$\begin{aligned}\pi(\nu_1\pi_1 + \nu_2\pi_2) &= \pi_1\nu_1\pi_1 + \pi_1\nu_2\pi_2 \\ &= 1_A\pi_1 + 0\pi_2 = \pi_1\end{aligned}$$

and $\pi_2(\nu_1\pi_1 + \nu_2\pi_2) = \pi_2$ similarly.

So $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$ since it's a factorisation of (π_1, π_2) through itself.

(iii) $\implies$ (i): given $\begin{tikzcd} & A \\ D \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$, if there exists $h : D \to C$ such that $\pi_1 h = f$, $\pi_2 h = g$ then

$$h = (\nu_1\pi_1 + \nu_2\pi - 2)h = \nu_1\pi_1 h + \nu_2\pi_2 h = \nu_1 f + \nu_2 g$$

But $\pi_1(\nu_1 f + \nu_2 g) = \pi_1\nu_1 f + \pi_1\nu_2 g = f + 0 = f$, and $\pi_2(\nu_1 f + \nu_2 g) = g$ similarly.

□

We say 0 is a **zero object** if it is both initial and terminal.

In (b) we call C a **biproduct** of A and B, and denote it $A \oplus B$.

Lemma 7.3. a. *A category with a zero object has a unique pointed structure.*

b. *If $\mathcal{C}$ is pointed with finite products and coproducts, and for every pair (A, B) the canonical $c : A + B \to A \times B$ defined by $\pi_i c \nu_j = \delta_{ij}$ is an isomorphism, then $\mathcal{C}$ has a unique semi-additive structure.*

Proof. a. We define $0 : A \to B$ to be the unique composite $A \to 0 \to B$, where 0 is the zero object (and this is the only possibility).

b. Given $(f, g) : A \rightrightarrows B$, we define $f +_L g$ to be the composite

$$A \xrightarrow{\binom{1}{1}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{(f,g)} B$$

and $f +_R g$ to be

$$A \xrightarrow{\binom{f}{g}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1,1)} B$$

Note that we have $h(f +_L g) = hf +_L hg$ and $(f +_R g)k = fk +_R gk$ when the composites are defined.

To show $f +_L 0 = f$, consider

$$\begin{tikzcd}
A \arrow[rr, "\binom{1}{1}"] \arrow[drrr, "1"'] & & A \times A \arrow[rr, "c^{-1}"] \arrow[dr, "\pi_1"] & & A + A \arrow[rr, "{(f,0)}"] \arrow[dl, "{(1,0)}"'] & & B \\
 & & & A \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

All three triangles commute, so $f +_L 0 = f$. Similarly $0 +_L f = f$ and dually $f +_R 0 = 0 +_R f = f$.

Now consider the composite

$$A \xrightarrow{\binom{1}{1}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1,1)} B$$

The composite $A \to B \times B$ is $\begin{pmatrix} f+_L g \\ h+_L k \end{pmatrix}$, so the whole is $(f+_L g)+_R(h+_L k)$. But it also equals $(f +_R h) +_L (g +_R k)$.

Now putting $g = h = 0$, we get $f +_R k = f +_L k$ so $+_R = +_L$.

Putting $f = k = 0$, we get $g + h = h + g$, so $+$ is commutative.

Putting $g = 0$, we get $f + (h + k) = (f + h) + k$, so $+$ is associative.

For uniqueness: given any semi-additive structure $+$, we must have $c^{-1} = \nu_1\pi_1 + \nu_2\pi_2$ by 7.2(b), so $+$ coincides with $+_L$ and $+_R$.

$\square$

Corollary 7.4. *If $\mathcal{C}$ and $\mathcal{D}$ are semi-additive with finite biproducts, then a functor $F : \mathcal{C} \to \mathcal{D}$ is additive $\iff$ it preserves finite (co)products.*

In particular, if F has an adjoint (on either side) then it's additive. Moreover, the adjunction is enriched over ***CMon****, in the sense that the bijection $\mathcal{D}(FA, B) \to \mathcal{C}(A, GB)$ is an isomorphism of commutative monoids.*

Definition 7.5. Let $A \xrightarrow{f} B$ be a morphism in a pointed category. The **kernel** of f is the equaliser $E \xrightarrow{\ker f} A$ of $A \underset{0}{\overset{f}{\rightrightarrows}} B$.

A monomorphism is **normal** if it occurs as a kernel.

In additive categories, every regular monomorphism is normal, since the equaliser of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is the kernel of $f - g$.

$A \xrightarrow{f} B$ is a **pseudo-monomorphism** if $\ker f$ is a zero map, i.e. if $fg = 0 \implies g = 0$. Again, in additive categories this holds if f is monic, but in semi-additive categories it's weaker.

Lemma 7.6. *In a pointed category with cokernels, every normal monomorphism is the kernel of its own cokernel.*

Proof. Suppose $f = \ker g$. Then $gf = 0$ so g factors through $\operatorname{coker} f$.

$$\begin{tikzcd} A \arrow[r, tail, "f"] & B \arrow[r, "g"] \arrow[dr, "\operatorname{coker} f"'] & C \\ E \arrow[ur, "h"] & & D \arrow[u] \end{tikzcd}$$

So if $h : E \to B$ satisfies $(\operatorname{coker} f)h = 0$, then $gh = 0$, and hence h factors uniquely through f. $\square$

In particular, if $\mathcal{C}$ has kernels and cokernels, then ker and coker induce a bijection between (isomorphism classes) of normal subobjects and normal quotients of any object.

Lemma 7.7. *If $\mathcal{C}$ is pointed with kernels and cokernels then $\mathcal{C}$ has images.*

Proof. Given $A \xrightarrow{f} B$, f factors through $\ker(B \xrightarrow{g} C)$

$$\begin{aligned} &\iff gf = 0 \\ &\iff g \text{ factors through } \operatorname{coker} f \\ &\iff \ker \operatorname{coker} f \text{ factors through } \ker g \end{aligned}$$

So $\ker \operatorname{coker} f$ is the smallest normal subobject of $\mathcal{C}$ through which f factors. $\square$

Definition 7.8. A category $\mathcal{A}$ is **ablelian** if

i. $\mathcal{A}$ is additive

ii. $\mathcal{A}$ has finite biproducts, kernels and cokernels (equivalently all finite limits and colimits)

iii. Every monomorphism is normal and very epimorphism is normal

Lemma 7.9. *In an additive category with finite biproducts, consider a square* $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$ *of objects and morphisms.*

The flatting of the square is the diagram

$$A \xrightarrow{\binom{f}{g}} B \oplus C \xrightarrow{(h,-k)} D$$

Then

i. the square commutes $\iff (h,-k)\binom{f}{g} = 0$

ii. the square is a pullback $\iff \binom{f}{g} = \ker(h,-k)$

iii. the square is a pushout $\iff (h,-k) = \operatorname{coker}\binom{f}{g}$

Proof. The composite of the flattening is $hf - kg$.

Then $\binom{f}{g}$ is universal among morphisms with $(h, -k)\binom{f}{g} = 0 \iff (f, g)$ is universal among pairs with $hf = kg$.

(iii) is dual to (ii). $\square$

Corollary 7.10. *In an abelian category, epimorphisms are stable under pullback.*

Proof. Suppose
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, two heads, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$
is a pullback with h epic.

Then $\binom{f}{g} = \ker(h, -k)$, but $(h, -k)$ is epic since h is, so $(h, -k) = \operatorname{coker}\binom{f}{g}$ and the square is a pushout.

Now suppose $C \xrightarrow{l} E$ satisfies $lg = 0$. Then the pair $(l, B \xrightarrow{0} E)$ factors through (k, h), say by $m : D \to E$, but then $mh = 0$ and so $m = 0$ since h is epic. So $l = mk = 0$. $\square$

Hence abelian categories are regular. In fact,

Theorem 7.11. *$\mathcal{A}$ is abelian $\iff$ $\mathcal{A}$ is additive and effective regular.*

Proof. Suppose that $\mathcal{A}$ is abelian. We've shown that $\mathcal{A}$ is regular, so we need to consider effectivity.

Let $R \overset{a}{\underset{b}{\rightrightarrows}} A$ be an equivalence relation. Form the pullback
$$\begin{tikzcd} K \arrow[r, "l"] \arrow[d, tail, "k"] & R \arrow[d, tail, "{(a,b)}"] \\ A \arrow[r, "\binom{1}{0}"] & A \oplus A \end{tikzcd}$$

k is monic, so it's the kernel of some $A \xrightarrow{f} B$. We'll show (a, b) is the kernel-pair of f.

Suppose $C \overset{x}{\underset{y}{\rightrightarrows}} A$ satisfy $fx = fy$, then $x - y$ factors as kz for some $z : C \to K$.

Now consider $lz + ry : C \to R$ where $r : A \to R$ satisfies $ar = br = 1_A$.

Now

$$\begin{aligned} a(lz + ry) &= alz + ary \\ &= kz + y \\ &= x - y + y = x \end{aligned}$$

$$\begin{aligned} b(lz + ry) &= blz + bry \\ &= 0z + y = y \end{aligned}$$

So $lz + ry$ is a factorisation of (x, y) through (a, b).

Conversely: first we show that any reflexive relation in an additive category (with finite limits) is symmetric and transitive: given $R \overset{a}{\underset{b}{\rightrightarrows}} A$, let $s = ra + rb - 1_R : R \to R$. Then

$$\begin{aligned} as &= ara + arb - a \\ &= a + b - a = b \end{aligned}$$

and $bs = a$ similarly.

Similarly, given the pullback $\begin{array}{ccc} T & \xrightarrow{q} & R \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle a} \\ R & \xrightarrow{b} & A \end{array}$, set $t = p + q - raq : T \to R$.

Then $at = ap + aq - araq = ap$ and $bt = bp + pq - brbp = bq$.

Now suppose $\mathcal{A}$ is effective regular and additive. Then $\mathcal{A}$ has finite biproducts and kernels.

Consider a monomorphism $K \overset{k}{\rightarrowtail} A$. Consider $K \oplus A \overset{(k,1)}{\underset{(0,1)}{\rightrightarrows}}$; this is jointly monic since if $(k,1)\binom{x}{y} = (0,1)\binom{x}{y} = 0$ then $y = 0$ and $kx = 0$, so $x = 0$.

It's reflexive with common splittling $A \xrightarrow{\binom{0}{1}} K \oplus A$. So it's an equivalence relation, and hence a kernel-pair of some $A \xrightarrow{f} B$. Now if $fg = 0$ for some $C \xrightarrow{g} A$, then $(g, 0)$ factors as

$$C \xrightarrow{\binom{u}{v}} K \oplus A \overset{(k,1)}{\underset{(0,1)}{\rightrightarrows}} A$$

whence $r = 0$, and $ku = g$. So $k = \ker f$.

Since $\mathcal{A}$ has images, it's enough to show that monomorphisms have cokernels. Given a monomorphism $K \overset{k}{\rightarrowtail} A$, form $K \oplus A \overset{(k,1)}{\underset{(0,1)}{\rightrightarrows}} A$; this is an equivalence relation, so has a coequaliser by 6.4. But a coequaliser for this pair is a cokernel fork.

Given an arbitrary epimorphism $A \overset{f}{\twoheadrightarrow} B$, we can factor it as $A \overset{q}{\to} I \overset{m}{\rightarrowtail} B$ where m is both regular monic and epic, hence an isomorphism, and q is a normal epimorphism. So f is normal. $\square$

Definition 7.12. a. Given a sequence

$$\dots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \dots$$

in a pointed category with kernels and cokernels, we say it's **exact at $\mathbf{C_n}$** if $\operatorname{im} f_{n+1} = \ker f_n$ (equivalently, $\operatorname{coker} f_{n+1} = \operatorname{coim} f_n$).

E.g.

i. $0 \longrightarrow A \xrightarrow{f} B$ exact $\iff$ f monic

ii. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ exact $\iff f = \ker g$

iii. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ exact $\iff f = \ker g$ and $g = \operatorname{coker} f$

b. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is **exact** if it preserves exact sequences (equivalently, preserves kernels and cokernels).

Note that a biproduct $A \oplus B$ is characterised by the exactness of

$$0 \longrightarrow A \xrightarrow{\binom{0}{1}} A \oplus B \xrightarrow{(0,1)} B \longrightarrow 0$$

plus the fact that $\binom{1}{0}$ and $(0, 1)$ are both split. So any exact functor preserves biproducts, and hence all finite limits and colimits.

c. F is called **left exact** if it preserves kernels (equivalently, exact sequences of type (ii)). This also implies additivity, and hence preservation of all finite limits.

Note that if F is left exact and preserves epimorphisms, then it's exact. In particular, a regular functor between abelian categories is exact.

Theorem 7.13. *Let $\mathcal{A}$ be a small abelian category. Then there exists a faithful (equivalently isomorphism-reflecting) exact functor $\mathcal{A} \to$ **AbGp** (and a full and faithful exact functor $\mathcal{A} \to \mathbf{Mod}_R$ for some ring R).*

Proof. As a regular category, $\mathcal{A}$ is totally supported, so by 6.9 there's a faithful regular functor $F = \operatorname{colim}_{J^{op}} \mathcal{A}(D(j), -) : \mathcal{A} \to \mathbf{Set}$.

If we regard the $\mathcal{A}(D(j), -)$ as functors $\mathcal{A} \to \mathbf{AbGp}$, we can take their colimit in $[\mathcal{A}, \mathbf{AbGp}]$, and the description of F in 6.9 still works. It's still regular, since $\mathbf{AbGp} \to \mathbf{Set}$ reflects finite limits and strong epimorphisms, so it's exact. It's still faithful.

To get the full and faithful functor to $\mathbf{Mod}_R$, we proceed as in 6.10, taking R to be the ring of endomorphisms of F in the additive category $[\mathcal{A}, \mathbf{AbGp}]$. $\square$

We can now proof all of the standard 'diagram chasing' lemmas for abelian categories by proving them in module categories, and then transferring them via the embedding theorem.

For example, we have the **Snake Lemma**:

Lemma 7.14. *Suppose given a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A_1 \arrow[r, red] \arrow[d] & A_2 \arrow[r, red] \arrow[d] & A_3 \arrow[d] \arrow[dddll, red] & \\
 & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[d] & \\
 & D_1 \arrow[r, red] \arrow[d] & D_2 \arrow[r, red] \arrow[d] & D_3 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in an abelian category where the black rows and columns are exact. Then there exist red morphisms forming an exact sequence.

Definition 7.15. A **complex** in an abelian category $\mathcal{A}$ is an infinite sequence

$$\dots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \dots$$

satisfying $f_n f_{n+1} = 0$ for all n (equivalently, $\ker f_n \geq \operatorname{im} f_{n+1}$).

The complexes in $\mathcal{A}$ form an abelian category $c\mathcal{A} = \mathbf{Add}(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ is the additive category with $\operatorname{ob}\mathcal{Z} = \mathbb{Z}$.

$$\begin{aligned} \mathcal{Z}(p,q) &= \mathbb{Z} \text{ if } q = p \text{ or } p = 1 \\ &= \{0\} \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} p \stackrel{m}{\to} q \stackrel{n}{\to} r &= mn \text{ if } p \geq q \geq r \geq p-1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Given a complex $C_\bullet$, we write

$$\begin{aligned} &Z_n(C_\bullet) \rightarrowtail C_n \text{ for } \ker f_n \\ &B_n(C_\bullet) \rightarrowtail C_n \text{ for } \operatorname{im} f_{n+1} \\ &Z_n(C_\bullet) \rightarrowtail H_n(C_\bullet) \text{ for } \operatorname{coker}(B_n \rightarrowtail Z_n) \end{aligned}$$

If we also write $C_n \to Q_n(C_\bullet)$ for $\operatorname{coker} f_{n+1}$, then we have a diagram

$$
\begin{tikzcd}
C_{n+1} \arrow[rrrr, "f_{n+1}"] \arrow[dr, two heads] & & & & C_n \arrow[rrrr, "f_n"] \arrow[dr, two heads] & & & & C_{n-1} \\
& Q_{n+1} \arrow[r, two heads] & B_n \arrow[r, tail] & Z_n \arrow[r, two heads] \arrow[ur, tail] & H_n \arrow[r, tail] & Q_n \arrow[r, two heads] & B_{n-1} \arrow[r, tail] & Z_{n-1} \arrow[ur, tail] &
\end{tikzcd}
$$

and $H_n = \mathrm{im}(Z_n \to Q_n)$. Note also that B_n, Q_n, H_n and Z_n are all (additive) functors $c\mathcal{A} \to \mathcal{A}$.

Theorem 7.16 (Mayer-Vietoris)**.** *Let*

$$0 \longrightarrow C_\bullet \longrightarrow D_\bullet \longrightarrow E_\bullet \longrightarrow 0$$

be a short exact sequence in $c\mathcal{A}$. Then there's an exact sequence

$$\ldots \longrightarrow H_n(C_\bullet) \longrightarrow H_n(D_\bullet) \longrightarrow H_n(E_\bullet) \longrightarrow H_{n-1}(C_\bullet) \longrightarrow H_{n-1}(D_\bullet) \longrightarrow \ldots$$

Proof. First apply the snake lemma to

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_n(C_\bullet) \arrow[r] \arrow[d] & Z_n(D_\bullet) \arrow[r] \arrow[d] & Z_n(E_\bullet) \arrow[d] & \\
0 \arrow[r] & C_n \arrow[r] \arrow[d] & D_n \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & D_{n-1} \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & 0 \\
& Q_{n-1}(C_\bullet) \arrow[r] \arrow[d] & Q_{n-1}(D_\bullet) \arrow[r] \arrow[d] & Q_{n-1}(E_\bullet) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

Then apply it to

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n(C_\bullet) \arrow[r] \arrow[d] & H_n(D_\bullet) \arrow[r] \arrow[d] & H_n(E_\bullet) \arrow[d] \arrow[dddll] & \\
 & Q_n(C_\bullet) \arrow[r] \arrow[d] & Q_n(D_\bullet) \arrow[r] \arrow[d] & Q_n(E_\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1}(C_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(D_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(E_\bullet) \arrow[d] & \\
 & H_{n-1}(C_\bullet) \arrow[r] \arrow[d] & H_{n-1}(D_\bullet) \arrow[r] \arrow[d] & H_{n-1}(E_\bullet) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

□

Definition 7.17. Let $f_\bullet, g_\bullet : C_\bullet \rightrightarrows D_\bullet$ be morphisms of complexes. A **homotopy** between $f_\bullet$ and $g_\bullet$ is a sequence of morphisms $h_n : C_n \to D_n$ such that $f_n - g_n = d_{n+1}h_n + h_{n-1}C_n$

$$
\begin{tikzcd}
C_{n+1} \arrow[r, "c_{n+1}"] & C_n \arrow[r, "c_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"] \\
D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1}
\end{tikzcd}
$$

We write $f_\bullet \simeq g_\bullet$ if there exists a homotopy. Note that $\simeq$ is an equivalence relation and it's a congruence (if $k_\bullet : D_\bullet \to E_\bullet$, then the composites $k_n h_n$ form a homotopy $k_\bullet f_\bullet \simeq k_\bullet g_\bullet$ and similarly on the other side).

So we can form the quotient category $c\mathcal{A}/\simeq$.

Lemma 7.18. *If $f_\bullet \simeq g_\bullet$ then $H_n(f_n) = H_n(g_n)$.*

Proof. The difference $Z_n(f_n) - Z_n(g_n)$ is the restriction to $Z_n(C_\bullet)$ of $d_{n+1}h_n$ since C_n restricted to Z_n is 0.

When we compose with the quotient map $Z_n(D_\bullet) \to H_n(D_\bullet)$, this term also becomes 0, so $H_n(f_n) - H_n(g_n) = 0$. □

Definition 7.19.

a. A category $\mathcal{C}$ has **enough projectives** if for all $A \in \operatorname{ob}\mathcal{C}$, there exists an epimorphism $P \twoheadrightarrow A$ with P projective.

   (Note that $\mathbf{Mod}_R$ has enough projectives, since free modules are projective.)

b. A **projective resolution** of an object A in an abelian category $\mathcal{A}$ is an exact sequence

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with all P_i projective.

Equivalently it's a complex $P_\bullet$ with $P_n = 0$ for all $n < 0$, P_n projective for all $n \geq 0$ and

$$H_n(P_\bullet) = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

If A has enough projectives, every object has a projective resolution: choose $P_0 \twoheadrightarrow A$ with P_0 projective. Let $K_0 \to P_0 = \ker(P_0 \twoheadrightarrow A)$, choose $P_1 \twoheadrightarrow K_0$ with P_1 projective, etc.

Lemma 7.20. *Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of A and B respectively. For any $f : A \to B$, there exists $g_\bullet : P_\bullet \to Q_\bullet$ with $H_0(g_\bullet) = f$ and any two such complex morphisms are homotopic.*

Proof. Consider

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r, "p_1"] \arrow[d, "g_1"] & P_0 \arrow[r, two heads, "p_0"] \arrow[d, "g_0"] & A \arrow[d, "f"] \\
\ldots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, two heads, "q_0"] & B
\end{tikzcd}$$

g_0 exists since P_0 is projective and q_0 epic. $q_0 g_0 p_1 = f p_0 p_1 = 0$ so $g_0 p_1$ factors through $\ker q_0 = \operatorname{im} q_1$, so g_1 exists since P_1 is projective and $\operatorname{coim} q_1$ is epic, etc.

Suppose we had another such morphism $g'_\bullet$. Then $q_0(g_0 - g'_0) = f p_0 - f p_0 = 0$. So $g_0 - g'_0$ factors through $\operatorname{im} q_1$, so $\exists h_0$ with $q_1 h_0 = g_0 - g'_0$.

Now

$$\begin{aligned} q_1(g_1 - g'_1 - h_0 p_1) &= g_0 p_1 - g'_0 p_1 - q_1 h_0 p_1 \\ &= (g_0 - g'_0 - q_1 h_0) p_1 = 0 \end{aligned}$$

So $g_1 - g'_1 - q_1 h_0$ factors through $\ker q_1 = \operatorname{im} q_2$, and hence there exists $h_1 : P_1 \to Q_2$ with $q_2 h_1 - g_1 - g'_1 - h_0 p_1$, etc. $\square$

Hence, we can regard the construction of projective resolutions as defining a functor $\mathcal{A} \to c\mathcal{A}/\simeq$.

Definition 7.21. Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. The **left derived functors** $L^n F$ of F are defined as follows: given A, let $P_\bullet$ be a projective resolution of A and define $L^n F(A) = H_n(FP_\bullet)$ for all $n \geq 0$.

This is well defined by 7.18 and 7.20: it's the composite

$$\mathcal{A} \xrightarrow{PR} c\mathcal{A}/\simeq \xrightarrow{cF/\simeq} c\mathcal{B}/\simeq \xrightarrow{H_n} B$$

(In fact, it's an additive functor.) If F is exact, then $L^0 F \cong F$ and $L^n F = 0$ for all $n > 0$.

If F is right exact (i.e. preserves cokernels), we still have $L^0 F \cong F$, since $FP_1 \to FP_0 \to FA \to 0$ is exact, but the $L^n F$ may be nonzero.

Theorem 7.22. *Let $\mathcal{A}, \mathcal{B}$ and F be as in 7.21. Then for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there is a long exact sequence*

$$\cdots \to L^1 FC \to L^0 FA \to L^0 FB \to L^0 FC \to 0$$

in $\mathcal{B}$ (so if F is right exact then $L^n F$ repair the lack of left exactness).

Proof. We show that if $P_\bullet$ and $R_\bullet$ are projective resolutions of A and C, then there's a projective resolution $Q_\bullet$ of B for which $Q_n = P_n \oplus R_n$ for all n and the morphisms $P_n \to Q_n \to R_n$ are $P_n \xrightarrow{\binom{1}{0}} P_n \oplus R_n \xrightarrow{(0,1)} R_n$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
P_1 \arrow[r, two heads] \arrow[d, "\binom{1}{0}"] & K_0 \arrow[r] \arrow[d] & P_0 \arrow[r, "p_0"] \arrow[d, "\binom{1}{0}"] & A \arrow[r] \arrow[d, "f"] & 0 \\
P_1 \oplus R_1 \arrow[r] \arrow[d, "(0,1)"] & L_0 \arrow[r] \arrow[d] & P_0 \oplus R_0 \arrow[r, "(fp_0,t)"] \arrow[d, "(0,1)"] & B \arrow[r] \arrow[d, "g"] & 0 \\
R_1 \arrow[r, two heads] & M_0 \arrow[r] \arrow[d] & R_0 \arrow[r, "r_0"] \arrow[d] \arrow[ur, "t"] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let t be such that $gt = r_0$. Suppose $x(fp_0, t) = 0$, then $xfp_0 = 0 \implies xf = 0$, so $x = yg$ for some y and $0 = xt = ygt = yr_0$, so $y = 0$. So $x = 0$.

Now $0 \to K_0 \to L_0 \to M_0 \to 0$ is exact by the Snake Lemma.

Hence, we can construct an epimorphism $P_1 \oplus R_1 \twoheadrightarrow L_0$ from the epimorphisms $P_1 \twoheadrightarrow K_0$ and $R_1 \twoheadrightarrow M_0$ as before.

Continue in the same way.

Since f is additive, it preserves the exactness of the columns $0 \to P_n \to P_n \oplus R_n \to R_n \to 0$, so the result follows from 7.16 applied to the exact sequence $0 \to FP_\bullet \to FQ_\bullet \to FR_\bullet \to 0$ in $\mathcal{B}$. $\square$